

Quasi-stationary simulation of the contact process

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Abstract

We review a recently devised Monte Carlo simulation method for the direct study of quasi-stationary properties of stochastic processes with an absorbing state. The method is used to determine the static correlation function and the interparticle gap-length distribution in the critical one-dimensional contact process. We also find evidence for power-law decay of the interparticle distance distribution in the two-particle subspace.

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I. INTRODUCTION

Stochastic processes with an absorbing state arise frequently in statistical physics [1,2], epidemiology [3] and related fields. Phase transitions to an absorbing state in spatially extended systems, exemplified by the contact process [4,5], are currently of great interest in connection with self-organized criticality [6], the transition to turbulence [7], and issues of universality in nonequilibrium critical phenomena [8–10].

The *quasi-stationary* (QS) distribution, (that is, conditioned on survival), is very useful in the study of processes with an absorbing state model. Conventional simulations of “stationary” properties of lattice models with an absorbing state actually study the quasi-stationary regime, given that the only true stationary state for a finite system is the absorbing one. We recently devised a simulation method that yields quasi-stationary properties directly [11]; it samples the QS probability distribution just as conventional Monte Carlo simulation samples the equilibrium distribution. Here we use the method to study the static correlation function and other configurational properties of the critical contact process, the prime example of an absorbing-state phase transition.

In the following section we review the basis of our method. Then in Sec. III we apply it to determine the static two-point correlation function of the contact process on a ring. We summarize our findings in Sec. IV.

II. BACKGROUND

Consider a continuous-time Markov process X_t taking values $n = 0, 1, 2, \dots, S$, with state $n=0$ absorbing. We use $p_n(t)$ to denote the probability that $X_t = n$, given some initial state X_0 . The survival probability is $P_s(t) = \sum_{n \geq 1} p_n(t) = 1 - p_0(t)$. We suppose that as $t \rightarrow \infty$ the p_n , normalized by the survival probability $P_s(t)$, attain a time-independent form, thus defining the quasi-stationary distribution \bar{p}_n :

$$\bar{p}_n \equiv \lim_{t \rightarrow \infty} \frac{p_n(t)}{P_s(t)}, \quad (n \geq 1), \quad (1)$$

with $\bar{p}_0 \equiv 0$. (We further assume that the limiting distribution does not depend on the initial state, as long as $X_0 \neq 0$.) The QS distribution is normalized so:

$$\sum_{n \geq 1} \bar{p}_n = 1. \quad (2)$$

As shown in [12], the QS distribution is the stationary solution to the following equation of motion (for $n > 0$)

$$\frac{dq_n}{dt} = -w_n q_n + r_n + r_0 q_n, \quad (3)$$

where $w_n = \sum_m w_{m,n}$ is the total rate of transitions out of state n , and $r_n = \sum_m w_{n,m} q_m$ is the flux of probability into this state. To see this, consider the master equation (Eq. (3) without the final term) in the QS regime. Substituting $q_n(t) = P_s(t) \bar{p}_n$, and noting that in the QS regime $dP_s/dt = -\bar{r}_0 = -P_s \sum_m w_{0,m} \bar{p}_m$, we see that the r.h.s. of Eq. (3) is identically zero if $q_n = \bar{p}_n$ for $n \geq 1$. The final term in Eq. (3) represents a redistribution of the probability r_0 (transferred to the absorbing state in the original master equation), to the nonabsorbing subspace. Each nonabsorbing state receives a share equal to its QS probability.

In [11] we introduce a process X_t^* , whose *stationary* probability distribution is the *quasi-stationary* distribution of X_t . (Note that in order to have a nontrivial stationary distribution, X_t^* cannot possess an absorbing state.) The probability distribution of X_t^* is governed by Eq. (3), which implies that for $n > 0$ (i.e., away from the absorbing state), the evolution of X_t^* is identical to that of X_t . When X_t enters the absorbing state, however, X_t^* instead jumps to a nonabsorbing one, and then resumes its “usual” evolution (with the same transition probabilities as X_t), until such time as another visit to the absorbing state is imminent.

In Eq. (3) the distribution q_n is used to determine the value of X_t^* when X_t visits the absorbing state. Although one has no prior knowledge of q_n (or its long-time limit, the QS distribution \bar{p}_n), one can, in a simulation, use the history X_s^* ($0 < s \leq t$) up to time t , to *estimate* the q_n . This is done by saving a sample $\{n_1, n_2, \dots, n_M\}$ of configurations visited. We update the sample by replacing from time to time one of the configurations with the current one. In this way the distribution for the process X_t^* will converge to the QS distribution (i.e., the stationary solution of Eq. (3)) at long times. Summarizing, the process X_t^* has the same dynamics as X_t , except when a transition to the absorbing state is imminent: X_t^* then is placed in a nonabsorbing state, selected at random from a sample over the history of the realization. (In the simulation, a list of M configurations is maintained. Whenever the time increases by 1, the list is updated with probability p_{rep} , by replacing a randomly chosen configuration on the list with the current one.) In effect, the final term in Eq. (3) is represented as a *memory* in the simulation.

The above scheme was shown [11] to yield precise results, in accord with the exact QS distribution for the contact process on a complete graph [12], and with conventional simulations of the same model on a ring, for which exact results are not available. QS simulation results were found rather insensitive to the choice of list size M and replacement rate p_{rep} . In the studies reported below we use $M = 1000$ and $p_{rep} = 0.001$.

III. CONTACT PROCESS: CORRELATION FUNCTION

The contact process (CP) [4] is a continuous-time Markov process on a lattice, in which each site i is either occupied ($\sigma_i(t) = 1$), or vacant ($\sigma_i(t) = 0$). Transitions from $\sigma_i = 1$ to $\sigma_i = 0$ occur at a rate of unity, independent of the neighboring sites. The reverse transition can only occur if at least one neighbor is occupied: the transition from $\sigma_i = 0$ to $\sigma_i = 1$ occurs at rate λr , where r is the fraction of nearest neighbors of site i that are occupied; thus the state $\sigma_i = 0$ for all i is absorbing. (λ is a control parameter governing the rate of spread of activity.) The order parameter $\rho = \langle \sigma_i \rangle$ is the fraction of occupied sites. The model exhibits a continuous phase transition at $\lambda_c = 3.297848(20)$ [13]. For $\lambda < \lambda_c$, the stationary value of ρ is zero.

The CP has attracted much interest as a prototype of a nonequilibrium critical point, a simple representative of the directed percolation (DP) universality class. Since its scaling properties have been discussed extensively [8–10] we review them only briefly. As the critical point is approached, the correlation length ξ and correlation time τ diverge, following $\xi \propto |\Delta|^{-\nu_\perp}$ and $\tau \propto |\Delta|^{-\nu_\parallel}$, where $\Delta = \lambda - \lambda_c$ is the distance from the critical point. The order parameter scales as $\rho \propto \Delta^\beta$ for $\Delta > 0$. At the critical point the quasi-stationary value of the order parameter scales as: $\rho \propto L^{-\beta/\nu_\perp}$.

An aspect of the CP that has not, to our knowledge, been studied in simulations is the static correlation function. Of particular interest is how correlations decay at the critical point. The correlation function is defined via

$$C(|i-j|) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \quad (4)$$

where the average is over the stationary distribution of the process.

Based on experience with equilibrium critical phenomena, we expect $C(r)$ to decay as a power law at the critical point. The decay exponent can be determined via a scaling argument. To begin, we normalize $C(r)$ to its value at $r = 0$:

$$c(r) \equiv \frac{C(r)}{C(0)} = \frac{C(r)}{\rho(1-\rho)} \quad (5)$$

Now consider the scaled variance

$$\chi = L^d \left(\langle \rho^2 \rangle - \rho^2 \right) \quad (6)$$

(L denotes the lattice size.) At the critical point, $\chi \sim L^{\gamma/\nu_\perp}$ with $\gamma = d\nu_\perp - 2\beta$ [8]. A simple calculation yields

$$\chi = L^{-d} \sum_{i,j} C(|i-j|) = \rho(1-\rho) \sum_{|\mathbf{x}| \leq L/2} c(|\mathbf{x}|) \quad (7)$$

where in the last step we used translation invariance, and assumed a hypercubic lattice of L^d sites. Now suppose $c(r) \sim r^{-\alpha}$ for large r . Approximating the sum by an integral, and recalling that $\rho \sim L^{-\beta/\nu_\perp}$ at the critical point, we find

$$\chi \sim L^{d-2\beta/\nu_\perp} \sim L^{d-\alpha-\beta/\nu_\perp} \quad (8)$$

implying that $\alpha = \beta/\nu_\perp$ and

$$C(r) \simeq \rho c(r) \propto (rL)^{-\beta/\nu_\perp} \quad (9)$$

in the critical stationary state. The relation $\alpha = \beta/\nu_\perp$ was demonstrated some time ago by Grassberger and de la Torre, who showed that $C(r) \sim r^{-2\delta/z}$ at the critical point [14]. (Their result is seen to be equivalent to ours when we recall the scaling relations $z = 2\nu_\perp/\nu_\parallel$ and $\delta = \beta/\nu_\parallel$.)

We use the QS simulation method outlined above to determine the correlation function. The process is simulated in five independent realizations of 2×10^8 time steps. As is usual, annihilation events are chosen with probability $1/(1+\lambda)$ and creation with probability $\lambda/(1+\lambda)$. A site is chosen from a list of currently occupied sites, and, in the case of annihilation, is vacated, while, for creation events, a nearest-neighbor site is selected at random and, if it is currently vacant, it becomes occupied. The time increment associated with each event is $\Delta t = 1/N_{occ}$, where N_{occ} is the number of occupied sites just prior to the attempted transition [8].

In Fig. 1 we plot $C^*(r) = L^{\beta/\nu_\perp} C(r)$ for $L = 1280$ and 2560 , using the best available estimate (from series analysis), $\beta/\nu_\perp = 0.252072(8)$ [15]. The data collapse for the two lattice sizes is nearly perfect. For $r \ll L$ the correlation function indeed follows a power law $C^* \sim r^{-\beta/\nu_\perp}$, while for $r = L/2$ it attains a minimum, as expected due to the periodic boundaries. To determine the decay exponent we analyze the local slope $\alpha(r)$ (see Fig. 2), obtained from a linear fit to the data for $\ln C^*$ versus $\ln r$, using points equally spaced in $\ln r$, in finite intervals $[r_0, 3r_0]$. For $r_0 \ll L$ the local slope is nearly constant, but it of course veers upward as r_0 approaches $L/2$. We therefore perform an extrapolation (to $r \rightarrow \infty$) of the local slope versus $1/r$, using only the data on which the results for the two lattice sizes agree, to eliminate finite-size effects. The result of this extrapolation is $\beta/\nu_\perp = 0.252(1)$, consistent with the best estimate.

For $\lambda < \lambda_c$, the correlation function decays exponentially, $C(r) \sim e^{-r/\xi}$. This is evident in the inset of Fig. 1, where we plot $\tilde{C} = r^{\beta/\nu_\perp} C(r)$. Exponential decay is clear for $\lambda = 0.99\lambda_c$; linear regression yields $\xi \simeq 356$ in this case, well below the system size ($L = 2560$) in this study. Interestingly, the decay of \tilde{C} is also perceptible for $\lambda = 0.999\lambda_c$, though partly masked due to the finite system size. In general the decay of correlations should be evident for $L > \xi$, where ξ is the correlation length in the infinite-size limit. Since $\xi \sim |\Delta|^{-\nu_\perp}$, deviations from criticality on the order of $|\Delta| \sim L^{-1/\nu_\perp}$ (or greater) should be detectable in the correlation function for a system of size L .

In the critical stationary state, the distribution of particles is scale-invariant, as reflected in the power-law decay of $C(r)$. The distribution of *gaps*, or strings of empty sites between successive occupied sites also follows a power law. (A gap of size g corresponds to sites i and $i + g + 1$ occupied and all intervening sites empty.) We determine the gap-size distribution $P(g)$ (normalized to the number of gaps in the configuration). As shown in Fig. 3, $P(g)$ exhibits a power law decay, $P \sim g^{-\tau}$, with $\tau \simeq 1.70$, over an intermediate range that appears to grow with system size. Analysis of the local slope yields $\tau = 1.73(1)$.

We can relate the exponent τ to other critical exponents via a simple scaling argument. Since there is one gap per particle, the mean gap size $\langle g \rangle$ is just the reciprocal of the particle density. Thus in a system of size L at the critical point, $\langle g \rangle \sim L^{\beta/\nu_\perp}$. Assuming $P(g) \sim g^{-\tau}$ for $g \geq 1$, we have

$$\langle g \rangle \sim \int_1^L g^{1-\tau} dg \sim L^{2-\tau} \sim L^{\beta/\nu_\perp} \quad (10)$$

implying $\tau = 2 - \beta/\nu_\perp \simeq 1.748$. Our simulation result is about 1% smaller than the value predicted by the scaling argument. The discrepancy is likely caused by finite-size corrections that limit the power-law regime of $P(g)$.

Finally, we report preliminary results on a surprising behavior of the interparticle distance in the two-particle subspace. Let d denote the separation between the occupied sites, given that exactly two sites are occupied. (We take the minimum distance under periodic boundaries, so that $d \leq L/2$.) Since particles are highly clustered in the critical CP, we should expect the two-particle distance distribution $P_2(d)$ to decay with separation. Our results (see Fig. 4) from QS simulations at the critical point suggest a power-law decay, $P_2(d) \sim d^{-\kappa}$, with $\kappa \simeq 2.45$. We have no way of relating this exponent to the known critical exponents. Indeed, whether $P_2(d)$ follows a power-law will have to be confirmed in larger-scale simulations. This is somewhat challenging since, as the system size increases, the probability of having exactly two particles becomes ever smaller.

IV. SUMMARY

We have applied a new simulation method for quasi-stationary properties to determine the static correlation function $C(r)$ of the critical contact process in one dimension. Our results support the behavior $C(r) \sim 1/(rL)^{\beta/\nu_\perp}$ anticipated from scaling arguments. We also studied the gap-size distribution, which shows evidence of a power-law decay, $P(g) \sim g^{-\tau}$, with $\tau \simeq 2 - \beta/\nu_\perp$, the value predicted by scaling. Finally we note an apparent scale-invariant behavior of the interparticle distance distribution $P_2(d)$ in the two-particle subspace. Study of the correlation function and the gap-size distribution promise to be useful in characterizing scaling behavior of new models, and may also be useful in locating the critical point.

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FIGURE CAPTIONS

FIG. 1. QS simulation results for the scaled correlation function $C^* = L^{\beta/\nu_\perp} C(r)$ in the critical one-dimensional contact process. Symbols: \times : $L = 1280$; $+$: $L = 2560$. The slope of the straight line is -0.252. Inset: semi-logarithmic plot of $\tilde{C} = r^{\beta/\nu_\perp} C(r)$ versus r in a system of 2560 sites, for $\lambda = \lambda_c$ (upper curve), 0.1% below λ_c (middle) and 1% below λ_c (lower).

FIG. 2. Local slope $-\alpha(r)$ of the correlation function versus $1/r$. Open symbols: $L = 1280$; filled symbols: $L = 2560$.

FIG. 3. Gap-length distribution in the critical one-dimensional CP. \square : $L = 640$; $+$: $L = 5120$. The slope of the straight line is -1.73.

FIG. 4. Distribution of interparticle distances d in the two-particle subspace of the critical CP. Open symbols: $L = 640$; filled symbols: $L = 1280$. The slope of the straight line is -2.45.







